MA 401 (001) HW8

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Problem 1a.

$$(a)\begin{cases} u_{tt} - u_{xx} = e^{-x} \\ u(x,0) = \cos(x), u_t(x,0) = 0 \end{cases}$$

Solution 1a.
We have the 1D wave equation with a source and nonzero initial data. Solve using the superposition principle

$$u = v + w,$$

where v solves the nonhomogeneous equation with $u(x,0) = 0$ and w solves the homogeneous equation with $f(x) = e^{-x}$.
The solution of w follows Duhamel's Formula:

$$w(x,t) = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$$

The solution of v is given by:

$$v = \frac{1}{2}(g(x-ct) + g(x+ct)) + \frac{1}{2c}\int_{x+ct}^{x-ct} h(y)\,dy$$

Then we have

$$w(x,t) = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{-y}\,dy\,ds$$

$$= \int_0^t [-e^{-y}]_{x-c(t-s)}^{x+c(t-s)}\,ds = \int_0^t [-e^{-(x+c(t-s))} + e^{-(x-c(t-s))}]\,ds = \int_0^t [e^()()]\,ds$$

Here we have $c = 1, f(x) = e^{-x}, g(x) = \cos(x)$, and $h(x) = 0$.
The solution is of the form

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) +$$

Problem 1b.

$$(a)\begin{cases} 4u_{tt} - 9u_{xx} = e^{-t} \\ u(x,0) = e^{-x}, u_t(x,0) = \cos(2x) \end{cases}$$

Solution 1b.
Problem 2a.

$$(a).\ F(s) = \frac{1}{s(s^2+4)}$$

Solution 2a.
First decompose $F(s)$ into $V(s)$ and $U(s)$:

$$F(s) = \frac{1}{s} \cdot \frac{1}{s^2+4}.$$

Which gives

$$V(s) = \frac{1}{s} \Rightarrow v(t) = 1;$$

$$U(s) = \frac{1}{s^2+4} \Rightarrow u(t) = \frac{1}{2}\sin(2t).$$

The convolution theorem tells us that

$$\mathcal{L}^{-1}\{V(s)U(s)\}(t) = (v * u)(t) = \int_0^t v(r)u(t-r)\,dr.$$

Hence,

$$\mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\{V(s)U(s)\}(t) = \int_0^t \frac{1}{2}\sin(2(t-r))\,dr.$$

Computing the above integral gives

$$\mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2}\Big[-\cos(2(t-r)) \cdot \frac{-1}{2}\Big]_0^t = \frac{1}{4}\Big[\cos(2t-2r)\Big]_0^t$$

$$= \frac{1}{4}\Big[\cos(2t-2t) - \cos(2t-0)\Big] = \boxed{\frac{1}{4} - \frac{1}{4}\cos(2t).}$$

Problem 2b.

$$(b).\ F(s) = \frac{1}{s^2(s+1)}$$

Solution 2b.
First decompose $F(s)$ into $V(s)$ and $U(s)$:

$$F(s) = \frac{1}{s^2} \cdot \frac{1}{s+1}.$$

Which gives

$$V(s) = \frac{1}{s^2} \Rightarrow v(t) = t;$$

$$U(s) = \frac{1}{s+1} \Rightarrow u(t) = e^{-t}.$$

The convolution theorem tells us that

$$\mathcal{L}^{-1}\{V(s)U(s)\}(t) = (v * u)(t) = \int_0^t v(r)u(t-r)\,dr.$$

Hence,

$$\mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\{V(s)U(s)\}(t) = \int_0^t re^{-(t-r)}\,dr.$$

Simplifying the above integral gives

$$\mathcal{L}^{-1}\{F(s)\}(t) = e^{-t}\cdot\int_0^t re^{r}\,dr = e^{-t}I.$$

Integration by parts: $u = r, du = dr; v = e^r, dv = e^r dr.$

$$I = \Big[re^r\Big]_0^t - \int_0^t e^r\,dr = \Big[te^t\Big] - \Big[e^t - 1\Big]$$

Then,

$$\mathcal{L}^{-1}\{F(s)\}(t) = e^{-t}\Big[te^t - e^t + 1\Big] = \boxed{e^{-t} + t - 1.}$$

Problem 2c.

$$(b).\ F(s) = \frac{1}{s^2(s^2+4)}$$

Solution 2c.

First decompose $F(s)$ into $V(s)$ and $U(s)$:

$$F(s) = \frac{1}{s^2}\cdot\frac{1}{s^2+4}.$$

Which gives

$$V(s) = \frac{1}{s^2} \Rightarrow v(t) = t;$$

$$U(s) = \frac{1}{s+1} \Rightarrow u(t) = \frac{1}{2}\sin(2t).$$

The convolution theorem tells us that

$$\mathcal{L}^{-1}\{V(s)U(s)\}(t) = (v * u)(t) = \int_0^t v(r)u(t-r)\,dr.$$

Hence,

$$\mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\{V(s)U(s)\}(t) = \frac{1}{2}\int_0^t r\cdot\sin(2(t-r))\,dr = \frac{I}{2}.$$

Integration by parts: $u = r, du = dr; v = \frac{1}{2}\cos(2t-2r), dv = \sin(2t-2r).$

$$I = \Big[r\cdot\frac{1}{2}\cos(2t-2r)\Big]_0^t - \int_0^t \frac{1}{2}\cos(2t-2r)\,dr$$

$$= \frac{t}{2}\cdot\cos(2t-2t) - 0 + \frac{1}{4}\Big[\sin(2t-2r)\Big]_0^t$$

$$\frac{t}{2} + \frac{1}{4}(\sin(2t-2t) - \sin(2t) = \frac{t}{2} - \frac{1}{4}\sin(2t).$$

$$\Rightarrow \mathcal{L}^{-1}\{F(s)\}(t) = \boxed{\frac{t}{4} - \frac{1}{8}\sin(2t),} \text{ since } \mathcal{L}^{-1}\{F(s)\}(t) = \frac{I}{2}.$$

Problem 3a.
Note: I solved this problem with partial fractions. There wasn't a requirement for convolution theorem on problem 3.

$$(a).\ 3u' = u - \cos(t), u(0) = 2$$

Solution 3a.

$$3u' = u - \cos(t)$$

$$\Longleftrightarrow$$

$$\mathcal{L}(3u') = \mathcal{L}(u - \cos(t))$$
$$3\mathcal{L}(u') = \mathcal{L}(u) - \mathcal{L}(\cos(t)$$
$$3(sU - 2) = U - \frac{s}{s^2+1}$$
$$U(3s-1) = 6 - \frac{s}{s^2+1}$$
$$\Rightarrow U = \frac{6s^2 - s + 6}{(3s-1)(s^2+1).}$$

Partial Fraction Decomposition:

$$\frac{A}{3s-1} + \frac{Bs+C}{s^2+1} = \frac{6s^2-s+6}{(3s-1)(s^2+1)}$$

$$\Longleftrightarrow$$

$$A(s^2+1) + (Bs+C)(3s-1) = 6s^2 - s + 6$$
$$As^2 + A + 3Bs^2 - C + 3Cs - Bs = 6s^2 - s + 6$$

This gives

$$\begin{cases} A + 3B = 6 \\ -B + 3C = -1 \\ A - C = 6 \end{cases} \Longleftrightarrow \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 6 \end{pmatrix}$$

Which has the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 6 \\ 0 & -1 & 3 & -1 \\ 1 & 0 & -1 & 6 \end{array}\right) \Longleftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 0 & 6 \\ 0 & -1 & 3 & -1 \\ 0 & -3 & -1 & 0 \end{array}\right) \Longleftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 0 & 6 \\ 0 & -10 & 0 & -1 \\ 0 & -3 & -1 & 0 \end{array}\right)$$

$$\Longleftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 0 & 6 \\ 0 & 1 & 0 & \frac{1}{10} \\ 0 & -3 & -1 & 0 \end{array}\right) \Longleftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{57}{10} \\ 0 & 1 & 0 & \frac{1}{10} \\ 0 & 0 & 1 & -\frac{3}{10} \end{array}\right) \Rightarrow \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} \frac{57}{10} \\ \frac{1}{10} \\ -\frac{3}{10} \end{pmatrix}$$

Rewrite U as

$$U = \frac{\frac{57}{10}}{3s-1} + \frac{\frac{1}{10}s - \frac{3}{10}}{s^2+1} = \frac{19}{10(s-\frac{1}{3})} + \frac{1}{10}\cdot\frac{s-3}{s^2+1} = \frac{19}{10}\cdot\frac{1}{s-\frac{1}{3}} + \frac{1}{10}\cdot\frac{s}{s^2+1} - \frac{3}{10}\cdot\frac{1}{s^2+1}$$

Recover $u(t)$ with Inverse Laplace Transform

$$\mathcal{L}^{-1}(U) = \frac{19}{10}\mathcal{L}^{-1}\Big(\frac{1}{s-\frac{1}{3}}\Big) + \frac{1}{10}\mathcal{L}^{-1}\Big(\frac{s}{s^2+1}\Big) - \frac{3}{10}\mathcal{L}^{-1}\Big(\frac{1}{s^2+1}\Big)$$

$$\Longleftrightarrow$$

$$u(t) = \boxed{\frac{19}{10}e^{\frac{t}{3}} + \frac{1}{10}\cos(t) - \frac{3}{10}\sin(t).}$$

Problem 3b.

$$(b).\ 2u' - 5u = 2e^{-2t}, u(0) = -1$$

Solution 3b.
U and V refer to $U(s)$ and $V(s)$, respectively.

$$2u' - 5u = 2e^{-2t}$$

$$\Longleftrightarrow$$

$$\mathcal{L}(2u' - 5u) = \mathcal{L}(v(t))$$

$$2(sU+1) - 5U = V$$

$$\Rightarrow U = \frac{V-2}{2s-5}$$

Applying Inverse Laplace Transform to both sides,

$$\mathcal{L}^{-1}(U) = u(t) = \mathcal{L}^{-1}(\frac{V}{2s-5} - \frac{2}{2s-5})$$

$$= \frac{1}{2}\mathcal{L}^{-1}(\frac{V}{s-\frac{5}{2}}) - \mathcal{L}^{-1}(\frac{1}{s-\frac{5}{2}}) = \frac{1}{2}\mathcal{L}^{-1}(V\cdot\frac{1}{s-\frac{5}{2}}) - e^{\frac{5}{2}t}.$$

By the convolution theorem, first term is reduced to

$$\frac{1}{2}\int_0^t v(r)u(t-r)\,dr = \frac{1}{2}\int_0^t 2e^{\frac{5}{2}r}\cdot e^{-2(t-r)}\,dr = e^{-2t}\int_0^t e^{\frac{9}{2}r}\,dr = e^{-2t}\cdot\frac{2}{9}[e^{\frac{9}{2}r}]_0^t = \frac{2}{9}e^{-2t+\frac{9}{2}t} - \frac{2}{9}e^{-2t}$$

Which gives the final solution

$$u(t) = \frac{2}{9}e^{\frac{5}{2}t} - \frac{2}{9}e^{-2t} - e^{\frac{5}{2}t} = \boxed{-\frac{7}{9}e^{\frac{5}{2}t} - \frac{2}{9}e^{-2t}}$$

Problem 3c.

$$(c).\ u' - u = e^{-t}, u(1) = 3$$

Solution 3c.
Start with a subsitution with $t = x + 1$ which gives
$u(t) = v(x+1) \Rightarrow u(1) = v(0) = 3$

$$v' - v = e^{-(x+1)}$$

$$\iff$$

$$\mathcal{L}(v' - v) = \mathcal{L}(e^{-(x+1)})$$
$$sV - 3 - V = W$$
$$V = \frac{W+3}{s-1} = \frac{W}{s-1} + \frac{3}{s-1}$$
$$v(t) = \mathcal{L}^{-1}(\frac{1}{s-1} \cdot W) + 3e^{x}$$

By the convolution theorem, the first term is reduced to

$$\int_0^x v(r)u(x-r)\, dr = \int_0^x e^{-(r+1)}e^{(x-r)}\, dr = e^{x-1}\int_0^x e^{-2r}\, dr = e[-\frac{e^{x-1}}{2}e^{-2r}]_0^x = \frac{-1}{2}e^{-x-1} + \frac{1}{2}e^{x-1}$$

Hence
$$v(t) = 3e^{x} - \frac{1}{2}e^{-x-1} + \frac{1}{2}e^{x-1}$$

Recover $u(t)$ with $x = t - 1$.

$$u(t) = \boxed{3e^{t-1} + \frac{1}{2}e^{t-2} - \frac{1}{2}e^{-t}}$$